On the surfaces associated with $\mathbb{C}P^{N-1}$ models

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Abstract. We study certain new properties of 2D surfaces associated with the $\mathbb{C}P^{N-1}$ models and the wave functions of the corresponding linear spectral problem. We show that $\mathfrak{su}(N)$-valued immersion functions expressed in terms of rank-1 orthogonal projectors are linearly dependent, but they span an $(N-1)$-dimensional subspace of the Lie algebra $\mathfrak{su}(N)$. Their minimal polynomials are cubic, except for the holomorphic and antiholomorphic solutions, for which they reduce to quadratic trinomials. We also derive the counterparts of these relations for the wave functions of the linear spectral problems. In particular, we provide a relation between the wave functions, which results from the partition of unity into the projectors. Finally, we show that the angle between any two position vectors of the immersion functions, corresponding to the same values of the independent variables, does not depend on those variables.

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1. Introduction

Over the past decades there has been a significant progress in the study of immersion of 2D surfaces in multidimensional Euclidean spaces obtained from $\mathbb{C}P^{N-1}$ models. The most fruitful approach to this subject has been achieved through the description of these surfaces in terms of the homogeneous variables f_k [2, 3, 13, 19] and orthogonal projectors P_k [18, 7, 6, 4, 8]. Using this language, we have established the recurrence relations for the projectors satisfying the $\mathbb{C}P^{N-1}$ model equations, for the wave functions of their spectral problems and consequently the immersion functions of 2D surfaces in the Lie algebra $\mathfrak{su}(N)$. In this paper we add certain new properties, concerning both the immersion functions and the wave functions, in order to enhance the algebraic and geometric characterization of the studied surfaces. The surfaces are defined by a contour integral [6]

$$X_k(\xi,\bar{\xi}) = i\int_\gamma \left(-[\partial P_k, P_k]d\xi + [\bar{\partial} P_k, P_k]d\bar{\xi}\right), \quad k = 0, ..., N-1, \tag{1.1}$$

which is independent of the path of integration $\gamma \in \mathbb{C}$ according to the dynamics of the orthogonal rank-1 projectors P_k. The projectors P_k, $0 \le k \le N-1$, are successive solutions [19] of the Euler-

Lagrange (E-L) equations in the form of a conservation law [18]

$$\partial\,[\bar{\partial}P, P] + \bar{\partial}\,[\partial P, P] = \mathbf{0}, \tag{1.2}$$

corresponding to the action integral

$$\int d\xi d\bar{\xi}\,\mathcal{L} = \mathrm{tr}(\partial P \cdot \bar{\partial} P) \tag{1.3}$$

with the constraint

$$P^2 = P. \tag{1.4}$$

Equation (1.2) ensures that the integrand of (1.1) is an exact differential. This mapping of an area on a Riemann sphere S^2 into a set of $\mathfrak{su}(N)$ matrices: $\Omega \ni (\xi, \bar{\xi}) \mapsto X_k(\xi, \bar{\xi}) \in \mathfrak{su}(N) \simeq \mathbb{R}^{N^2-1}$ is a generalised Weierstrass formula for immersion (GWFI) of 2D surfaces in the Euclidean space $\mathbb{R}^{N^2-1}$ [9, 10, 12]. The target spaces of the projectors P_k are one-dimensional vector functions $f(\xi, \bar{\xi}) \in \mathbb{C}^N$, constituting an orthogonal basis in $\mathbb{C}^N$ [2, 19]

$$P_k = [1/(f_k^\dagger \cdot f_k)] f_k \otimes f_k^\dagger, \tag{1.5}$$

All the projectors may be obtained from the first projector P_0, whose target space is an arbitrary holomorphic vector function $f_0(\xi)$, by the recurrence formulae derived in our previous work [4]. The projectors are orthogonal to each other and they constitute a partition of unity [2, 19]

$$P_k P_l = \delta_{kl} P_k \qquad \text{(no summation)}, \tag{1.6}$$

$$\sum_{k=0}^{N-1} P_k = \mathbb{I}, \tag{1.7}$$

where $\mathbb{I}$ is the N-dimensional unit matrix.

For the surfaces corresponding to the projectors P_k, the integration may be performed explicitly, with the result [6]

$$X_k = -i\left(P_k + 2\sum_{j=0}^{k-1} P_j\right) + ic_k\mathbb{I}, \quad c_k = \frac{1+2k}{N}, \quad 0 \leq k \leq N-1. \tag{1.8}$$

In this paper we use the inverse formulae for the projectors in terms of the surfaces X_k, obtained in our previous work [4]

$$P_k = {X_k}^2 - 2i\,(c_k - 1)\,X_k - c_k\,(c_k - 2)\,\mathbb{I}. \tag{1.9}$$

to turn the projective property, the partition of unity and the orthogonal property into corresponding properties of the surfaces. Namely, we obtain the minimal polynomials, dimensionality of the spanned subspace of $\mathbb{R}^{N^2-1}$, and the angle between the position vectors of the surfaces, respectively.

In a similar manner we obtain the inverse formulae for the projectors in terms of the wave functions and the spectral parameter of the spectral problems. These relations allow us to derive the corresponding relations for the wave functions. In particular, we determine the minimal polynomial and a linear-dependence equation for those functions.

2. Projectors and soliton surfaces

In order to obtain the relations between surfaces X_k, it is convenient to express them in terms of the orthogonal rank-1 projectors P_k. Making use of the expression (1.8) and the partition of unity in terms of the projectors (1.7), it can be shown that the algebraic condition

$$\sum_{k=0}^{N-1}(-1)^k X_k = \mathbf{0} \tag{2.1}$$

holds. This means that the $\mathfrak{su}(N)$-valued immersion functions X_j are linearly dependent.
The equation (2.1) follows directly from the GWFI (1.8) in terms of projectors P_k and the decomposition of unity (1.7). Indeed, subtracting (1.8) for neighbouring k, we obtain

$$X_k - X_{k-1} = -i(P_k + P_{k-1}) + \frac{2i}{N}\mathbb{I}. \tag{2.2}$$

By adding the equations (2.2) for every second k, we obtain an equation containing the sum of all the projectors P_k, which is the unit matrix, according to (1.7). The final result proves to be exactly eq. (2.1).

Note that we can obtain the projectors P_k from the surfaces X_k not only as quadratic functions of the surfaces (1.9), but also as linear combinations of the surfaces $X_0, \ldots, X_k$ [4]

$$P_k = i\sum_{j=1}^{k}(-1)^{k-j}\left(X_j - X_{j-1}\right) + (-1)^k iX_0 + \frac{1}{N}\mathbb{I}. \tag{2.3}$$

Thus we may regain all the projectors $P_0, ..., P_{N-1}$ from the appropriate linear combinations of the surfaces $X_0, ..., X_{N-1}$ and the unit matrix. This means that these surfaces span an $(N-1)$ dimensional subspace of the $\mathfrak{su}(N)$ Lie algebra, as the projectors are linearly independent.

The projective property $P_k^2 = P_k$ imposes an algebraic constraint on the surfaces X_k. To find the lowest order constraint on X_k, we compare $P_k \cdot X_k$ obtained from (1.9) multiplied X_k with $P_k \cdot X_k$ obtained from (1.8) multiplied by P_k. This yields a cubic matrix equation

$$(X_k - ic_k\mathbb{I})[X_k - i(c_k - 1)\mathbb{I}][X_k - i(c_k - 2)\mathbb{I}] = \mathbf{0}, \quad 0 < k < N-1. \tag{2.4}$$

For holomorphic ($k = 0$) and antiholomorphic ($k = N-1$) solutions of the $\mathbb{C}P^{N-1}$ equation (1.2) the minimal polynomial for the matrix-valued functions X_k is quadratic. Namely, for the surfaces corresponding to the holomorphic solutions we have

$$(X_0 - ic_0\mathbb{I})[X_0 - i(c_0 - 1)\mathbb{I}] = \mathbf{0}, \quad k = 0 \tag{2.5}$$

and, using $c_0 + c_{N-1} = 2$, we get

$$(X_{N-1} + ic_0\mathbb{I})[X_{N-1} + i(c_0 - 1)\mathbb{I}] = \mathbf{0}, \quad k = N-1 \tag{2.6}$$

for the antiholomorphic ones. Although equation (2.6) is apparently the Hermitian conjugate of the equation (2.5), their solutions do not have to be the Hermitian conjugates of each other.

Condition (2.4), as well as two other conditions (2.5) and (2.6), have simple interpretation if we diagonalize them (which is always possible, as the matrices are anti-Hermitian). It follows that the following numbers are eigenvalues of X_k $k = 1, ..., N-2$

$$ic_k, \quad i(c_k - 1) \quad \text{and} \quad i(c_k - 2), \tag{2.7}$$

while only the first two are eigenvalues for $k = 0$ and only the last two for $k = N-1$.

The three values listed in (2.7) are the only eigenvalues of X_k. More precisely

- The non-degenerate eigenvalue $i\,(c_k-1)$ occurs at every X_k, $\quad k=0,...,N-1$; the corresponding eigenvector is f_k (the latter follows directly from (1.8) and from the fact that $f_l,\ l=0,...,N-1$ are eigenvectors of the projectors P_k with the eigenvalue δ_{kl}).
- The k-fold degenerate eigenvalue $i\,(c_k-2)$ occurs at every X_k, except for $k=0$; the k corresponding eigenvectors are $f_0,...,f_{k-1}$.
- The $(N-1-k)$-fold degenerate eigenvalue ic_k occurs at every X_k, except for $k=N-1$; the corresponding eigenvectors are $f_{k+1},...,f_{N-1}$.

Equation (2.4), together with (2.5) and (2.6), constitute the lowest degree constraints on the immersion functions X_k of the surfaces (direct substitution of (1.9) into the projective property would yield a 4^{th} degree one). Although equation (2.4) is obvious when we look at the source of X_k (1.8), it is nevertheless a nontrivial constraint imposed on the surfaces. Since all the eigenvalues are independent of the coordinates $(\xi,\bar{\xi})$, the whole kinematics of a moving frame may only be due to variation of the diagonalizing (unitary) matrix.

Let us now present certain geometrical aspects of surfaces immersed in the $\mathfrak{su}(N)$ Lie algebras. Once we have the immersion functions of the surfaces, we can describe their metric and curvature properties.

(i) Let g_k be the metric tensor corresponding to the surface X_k. Its components will be marked with indices outside the parentheses to distinguish them from the index of the surface. Then the diagonal elements of the metric tensor are zero. This property directly follows from vanishing of $\text{tr}(\partial P_k \partial P_k)$ and its Hermitian conjugate, proven in [4]

$$\begin{aligned}(g_k)_{11}&=(\partial X_k,\partial X_k)=\frac{1}{2}\text{tr}([\partial P_k,P_k]\cdot[\partial P_k,P_k])=-\frac{1}{2}\text{tr}(\partial P_k\cdot\partial P_k)=0,\\(g_k)_{22}&=(\bar{\partial} X_k,\bar{\partial} X_k)=\frac{1}{2}\text{tr}([\bar{\partial} P_k,P_k]\cdot[\bar{\partial} P_k,P_k])=-\frac{1}{2}\text{tr}(\bar{\partial} P_k\cdot\bar{\partial} P_k)=0,\end{aligned}\tag{2.8}$$

where the inner product (A,B) of the $\mathfrak{su}(N)$ matrices is defined by [5]

$$(A,B)=-\frac{1}{2}\text{tr}(A\cdot B).\tag{2.9}$$

(ii) The nonzero off-diagonal element $(g_k)_{12}=(g_k)_{21}$ is equal to

$$(g_k)_{12}=-\frac{1}{2}\text{tr}(\partial X_k\cdot\bar{\partial} X_k)=-\frac{1}{2}\text{tr}([\partial P_k,P_k]\cdot[\bar{\partial} P_k,P_k])=\frac{1}{2}\text{tr}(\partial P_k\cdot\bar{\partial} P_k).\tag{2.10}$$

Thus the first fundamental form reduces to

$$I_k=\text{tr}(\partial P_k\cdot\bar{\partial} P_k)\,d\xi d\bar{\xi}.\tag{2.11}$$

The second fundamental form

$$II_k=(\partial^2 X_k-(\Gamma_k)^1_{11}\partial X_k)d\xi^2+2\partial\bar{\partial} X_k d\xi d\bar{\xi}+(\bar{\partial}^2 X_k-(\Gamma_k)^2_{22}\bar{\partial} X_k)d\bar{\xi}^2,\tag{2.12}$$

is easy to find when we determine the Christoffel symbols $(\Gamma_k)^1_{11}$ and $(\Gamma_k)^2_{22}$. These are the only nonzero components of Γ_k. From equation (2.10), we get

$$(\Gamma_k)^1_{11}=\partial\ln(g_k)_{12},\quad(\Gamma_k)^2_{22}=\bar{\partial}\ln(g_k)_{12}.\tag{2.13}$$

Using (1.1) and the E-L equations (1.2) together with (2.13), we can write (2.12) as

$$\begin{aligned} II_k = & -\mathrm{tr}(\partial P_k \cdot \bar{\partial} P_k)\, \partial \frac{[\partial P_k, P_k]}{\mathrm{tr}(\partial P_k \cdot \bar{\partial} P_k)} d\xi^2 + 2i\, [\bar{\partial} P_k, \partial P_k] d\xi d\bar{\xi} \\ & + \mathrm{tr}(\partial P_k \cdot \bar{\partial} P_k)\, \bar{\partial} \frac{[\bar{\partial} P_k, P_k]}{\mathrm{tr}(\partial P_k \cdot \bar{\partial} P_k)} d\bar{\xi}^2. \end{aligned} \tag{2.14}$$

Implementation of the above result for the metric of the surfaces induced by Veronese solutions of the E-L equations (1.2) is presented in detail in [4].

Also the following 2^{nd} order differential conditions hold:

$$(\partial\bar{\partial} X_k, \partial X_k) = 0, \qquad (\partial\bar{\partial} X_k, \bar{\partial} X_k) = 0, \tag{2.15}$$

$$(\partial\bar{\partial} X_k, \partial^2 X_k) = 0, \qquad (\partial\bar{\partial} X_k, \bar{\partial}^2 X_k) = 0. \tag{2.16}$$

Equations (2.15) follow from direct differentiation of (2.8). Hence, the mixed derivatives of the matrices X_k coincide and are normal to the surfaces [5]. The second order differential constraints (2.16) are calculated straightforwardly from the definition

$$\begin{aligned} (\partial\bar{\partial} X_k, \partial^2 X_k) = & -\frac{1}{2}\mathrm{tr}([\bar{\partial} P_k, \partial P_k] \cdot [\partial^2 P_k, P_k]) = -\frac{1}{2}\big[\mathrm{tr}(\bar{\partial} P_k \partial P_k \partial^2 P_k P_k) \\ & - \mathrm{tr}(\partial P_k \bar{\partial} P_k \partial^2 P_k P_k) - \mathrm{tr}(\bar{\partial} P_k \partial P_k P_k \partial^2 P_k) + \mathrm{tr}(\partial P_k \bar{\partial} P_k P_k \partial^2 P_k)\big] = 0, \end{aligned} \tag{2.17}$$

since the conditions

$$P_k \bar{\partial} P_k \partial P_k = \bar{\partial} P_k \partial P_k P_k, \qquad P_k \partial P_k \bar{\partial} P_k = \partial P_k \bar{\partial} P_k P_k, \tag{2.18}$$

hold. Similarly, the second relation in (2.16) holds for its respective Hermitian conjugates. Note that equations (2.15) and (2.16) are gauge-invariant since they are expressed in terms of the projectors P_k.

We now show that the surfaces X_k, X_l do not have common points for $k \neq l$, with the exception of X_0 and X_1 in the $\mathbb{C}P^1$ model, where simply X_0 coincides with X_1.

Indeed, let $l > k$ be two different indices of the surfaces. Subtracting (1.8) from the analogous expression for X_l, we obtain

$$-i \left[P_l - P_k + 2\sum_{j=k}^{l-1} P_j - \frac{2(l-k)}{N}\mathbb{I} \right] = \mathbf{0} \tag{2.19}$$

Multiplying eq. (2.19) by P_k, we obtain

$$P_k \left[1 - \frac{2(l-k)}{N} \right] = \mathbf{0}. \tag{2.20}$$

On the other hand, when we multiply both hand sides of (2.19) by P_{l-1}, we get

$$P_{l-1} \left[1 - \frac{l-k}{N} \right] = \mathbf{0} \quad \text{for } k < l-1, \text{ and} \tag{2.21}$$

$$P_k \left(1 - \frac{2}{N} \right) = \mathbf{0} \quad \text{for } k = l-1. \tag{2.22}$$

All of the equations (2.20), (2.21) and (2.22) may only be satisfied when $N = 2,\ l = 1,\ k = 0$. In that case (2.1) yields immediately $X_1 = X_0$.

We now show that the immersion functions $X_k,\ X_m$ make a constant angle in the sense of the Euclidean inner product (2.9), i.e. the angle Φ_{km} between the immersion functions X_k and X_m does not depend on the particular choice of projector P_0, nor on the coordinates $\xi,\ \bar{\xi}$. Namely, the angle Φ_{km} between two different functions $X_k,\ X_m,\ k < m$, is given by the equation

$$\cos\Phi_{km} = \frac{c_k(2-c_m)}{\{[c_k(2-c_k)-1/N]\,[c_m(2-c_m)-1/N]\}^{1/2}} \tag{2.23}$$

The formula (2.23) may be obtained in a straightforward way either by calculating the appropriate scalar product from the GWFI for $X_k,\ X_m$ (1.8) (bearing in mind that the projectors $P_0,\ P_1,\ \ldots,\ P_k$ are mutually orthogonal), or by direct operations on the eigenvalues of the immersion functions. In either way we obtain for $m > k$

$$(X_k, X_m) = -(1/2)\,\mathrm{tr}(X_k\cdot X_m) = (N/2)\,c_k(2-c_m), \tag{2.24}$$

while

$$(X_k, X_k) = -(1/2)\,\mathrm{tr}(X_k\cdot X_k) = (1/2)\,[N\,c_k(2-c_k)-1]. \tag{2.25}$$

It may easily be proven that (2.23) always yields $\cos\Phi_{km} \in (0,1)$ unless $N = 2$ (and obviously $k = 0,\ m = 1$), for which the surfaces coincide (and the cosine is obviously equal to 1). Equation (2.23) is symmetric with respect to a transformation $k \longleftrightarrow N-1-m$, which may be seen e.g. in the table of $\cos\Phi_{km}$ for the $\mathbb{C}P^3$ model

$k\setminus m$	1	2	3
0	$5/\sqrt{33}$	$\sqrt{3/11}$	$1/3$
1		$9/11$	$\sqrt{3/11}$
2			$5/\sqrt{33}$

We can regard the immersion functions X_k as position vectors, whose ends draw the two-dimensional surfaces in a N^2-1-dimensional $\mathfrak{su}(N)$ algebra. The above result means that the position vectors make a constant angle with each other, independent of the variables $\xi,\ \bar{\xi}$. Moreover, the angle is the same for all choices of the P_0 solutions of the E-L equations (1.2) within a particular $\mathbb{C}P^{N-1}$ model.

3. Projectors and the spectral problem

The spectral problem is closely related to the immersion functions of the surfaces. The relation between the wave functions and the immersion functions is given by the Sym-Tafel (ST) formula [14, 15, 16, 17]. The wave functions are also related to the immersion functions by their asymptotic properties. No wonder that the results of the previous section have their counterparts in corresponding relations between the wave functions.

Similarly to the surfaces, the wave functions of the spectral problem can also be expressed in terms of the projectors. The spectral problem found by Zakharov and Mikhailov [18] reads

$$\partial\Phi_k = \frac{2}{1+\lambda}[\partial P_k, P_k]\Phi_k, \qquad \bar{\partial}\Phi_k = \frac{2}{1-\lambda}[\bar{\partial}P_k, P_k]\Phi_k, \qquad k = 0,1,\ldots,N-1, \tag{3.1}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. An explicit solution of the linear spectral problem (3.1) for which Φ_k tends to $\mathbb{I}$ as $\lambda \to \infty$ is given by [1]

$$\Phi_k = \mathbb{I} + \frac{4\lambda}{(1-\lambda)^2}\sum_{j=0}^{k-1} P_j - \frac{2}{1-\lambda}P_k, \quad {\Phi_k}^{-1} = \mathbb{I} - \frac{4\lambda}{(1+\lambda)^2}\sum_{j=0}^{k-1} P_j - \frac{2}{1+\lambda}P_k, \tag{3.2}$$

where λ is purely imaginary and thus $\Phi_k \in SU(N)$. This in turn yields the projectors P_k in terms of the wave functions [4]

$$P_k = (1/4)\left[2(1+\lambda^2)\mathbb{I} - (1-\lambda)^2\Phi_k - (1+\lambda)^2\Phi_k^{-1}\right]. \tag{3.3}$$

The projective property of P_k may be represented in terms of Φ_k as a factorisable 4th degree expression with one double (squared) factor

$$P_k^2 - P_k = (1/16)\Phi_k^{-2}(\mathbb{I} - \Phi_k)\left[(1+\lambda)^2 - (1-\lambda)^2\Phi_k\right]\left[(1+\lambda) - (1-\lambda)\Phi_k\right]^2 = \mathbf{0}. \tag{3.4}$$

Hence, the minimal polynomials of the matrices Φ_k are cubic and they satisfy the equation, resembling the corresponding equation for the surfaces X_k (2.4), but explicitly depending on the spectral parameter

$$(\mathbb{I} - \Phi_k)\left[(1+\lambda)\mathbb{I} - (1-\lambda)\Phi_k\right]\left[(1+\lambda)^2\mathbb{I} - (1-\lambda)^2\Phi_k\right] = \mathbf{0} \tag{3.5}$$

for $1 < k < N-1$. Similarly to the surfaces (2.5), (2.6), quadratic matrix equations are sufficient for $k = 0$ and $k = N-1$: the equation with only the first two factors of (3.5) is satisfied by Φ_0 and that with only the last two factors of (3.5) by Φ_{N-1}.

The immersion functions X_k may be expressed in terms of the wave functions Φ_k in two ways: either by the ST formula [14, 15, 16, 17]

$$X_k^{ST} = -\frac{i}{2}(1-\lambda^2)\Phi_k^{-1}\partial_\lambda\Phi_k \tag{3.6}$$

or as a limit [4]

$$X_k = i\lim_{\lambda\to\infty}\left[\frac{\lambda}{2}\Phi_k + \left(c_k - \frac{\lambda}{2}\right)\mathbb{I}\right]. \tag{3.7}$$

Using equation (3.7), one can check by explicit calculation that the cubic minimal polynomial (3.5) for the wave function Φ_k coincides with the cubic polynomial (2.4) for the immersion function X_k in the limit $\lambda \to \infty$.

We now show that the partition of unity (1.7) for the projectors P_k imposes constraints on the wave functions Φ_k, given by the relation

$$\sum_{j=0}^{N-1}\Phi_j\left(\frac{1-\lambda}{1+\lambda}\right)^j = \frac{1+\lambda}{2\lambda}\left[1 - \left(\frac{1-\lambda}{1+\lambda}\right)^{N-2}\right]\mathbb{I}. \tag{3.8}$$

Indeed, using the wave functions Φ_k, which can be expressed in terms of the projectors P_k through the formula (3.2), for the indices k and $k-1$, we obtain

$$\sum_{k=0}^{N-1}\sum_{j=0}^{k}(\Phi_{j-1} - \Phi_j)\left(\frac{1+\lambda}{1-\lambda}\right)^{k-j} = \frac{2}{1-\lambda}\mathbb{I}, \tag{3.9}$$

or equivalently

$$\sum_{j=0}^{N-1}(\Phi_{j-1} - \Phi_j)\left(\frac{1+\lambda}{1-\lambda}\right)^{N-j} - \sum_{j=0}^{N-1}(\Phi_{j-1} - \Phi_j) = \frac{4\lambda}{1-\lambda^2}\mathbb{I}, \tag{3.10}$$

or, factoring out the coefficients with respect to Φ_{j-1} and Φ_j, we get

$$\sum_{j=0}^{N-1}\Phi_{j-1}\left(\frac{1+\lambda}{1-\lambda}\right)^{N-j} - \sum_{j=0}^{N-1}\Phi_j\left(\frac{1+\lambda}{1-\lambda}\right)^{N-j} - \Phi_{N-1} = \left(\frac{1+\lambda}{1-\lambda}\right)^2\mathbb{I}. \tag{3.11}$$

Finally, this expression can be written in the form

$$\sum_{j=0}^{N-1} \Phi_j \left(\frac{1-\lambda}{1+\lambda}\right)^j = \frac{1+\lambda}{2\lambda}\left[1-\left(\frac{1-\lambda}{1+\lambda}\right)^{N-2}\right]\mathbb{I}. \tag{3.12}$$

Equation (3.12), transformed into an appropriate equation for the variable $(\lambda/2)\Phi_j-(c_j-\lambda/2)\mathbb{I}$, turns into equation (3.7) in the limit $\lambda\to\infty$.
Note that in view of the linear dependence of the immersion functions X_k, i.e. equation (2.1), the ST formula (3.6) leads to the following differential constraint on the wave functions Φ_k

$$\sum_{j=0}^{N-1}(-1)^j X_j^{ST} = -\frac{i}{2}(1-\lambda^2)\partial_\lambda \ln \prod_{j=0}^{N-1}\Phi_j^{(-1)^j} = \mathbf{0}. \tag{3.13}$$

This implies that the expression $\prod_{2l<N}\Phi_{2l}\prod_{2l+1<N}\Phi_{2l+1}^{-1}$ is independent of the spectral parameter λ but it may depend on the variables ξ and $\bar{\xi}\in\mathbb{C}$.

4. Concluding remarks

In our work we develop the approach proposed in [4], which relies on construction of the consecutive surfaces and immersion functions in terms of projectors. The inverse formulae found in that work allowed for deriving additional properties of the surfaces immersed in the $\mathfrak{su}(N)$ Lie algebra and functions immersed in the $SU(N)$ Lie group. In particular

- We have shown that the number of linearly independent surfaces, associated with the $\mathbb{C}P^{N-1}$ models is $N-1$.
- The angles between the position vectors of any two surfaces are constant (independent of ξ, $\bar{\xi}$ and independent of the choice of the holomorphic solution). The surfaces do not intersect with each other for $\mathbb{C}P^{N-1}$, $N\geq 2$; the only two surfaces of $\mathbb{C}P^1$ coincide.
- All the surfaces associated with the $\mathbb{C}P^{N-1}$ models satisfy a 3rd-degree matrix equation, which reduces to a 2nd-degree equation for the holomorphic and antiholomorphic solutions of the E-L equations (1.2).
- The corresponding relations hold for the wave functions of the spectral problem. Moreover the asymptotic properties of those functions while the spectral parameter tends to infinity connect the relations for the wave functions with those for the immersion functions.

The proposed approach opens a field of further research for other sigma models.

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